UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-98**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

98-C0976-3-160

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All questions may be attempted but only marks obtained on the best **five** *solutions will count. The use of an electronic calculator is* **not** *permitted during this examination.*

1. Let f be a real-valued function defined on some open interval (a, b). If $c \in (a, b)$, explain what is meant by the statement "f is differentiable at c".

   By using this definition show that, if $f(x) = x^n$, n a non-negative integer, then f is differentiable at c with derivative nc^{n-1}.

   Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence R.

   Show that, for $|x| < R$, g is continuous at x.

2. State and prove each of Rolle's Theorem, the Cauchy Mean Value Theorem and L'Hôpital's Rule.

   Evaluate

   (i) $\lim_{x \to 0} \dfrac{2x + x^2 + 2\log(1-x)}{x^3}$,

   (ii) $\lim_{x \to 0+} x(\log x)^2$,

   (iii) $\lim_{x \to \frac{\pi}{2}} \left(\dfrac{\pi}{2} - x\right)^2 \tan^2 x$.

3. (i) Without using any general results, show that, if $a_n = \left(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}\right) - \log n$, the sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded below.

   Hence show that $a_n \to a$ as $n \to \infty$, where $0 \le a \le 1$.

   (ii) Given that

   $$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

   and that a power series can be differentiated term by term within its radius of convergence, show that

   $$\cos(x+y) = \cos x \cos y - \sin x \sin y, \forall x, y \in \mathbb{R}.$$

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4. (i) Let f be a bounded real-valued function on $[a,b]$ and suppose that, for each $\varepsilon > 0$, there exists a partition P such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Prove that f is Riemann integrable on $[a,b]$.

(ii) If f is a continuous function on $[a,b]$ show that f is uniformly continuous on $[a,b]$. Hence prove that f is Riemann integrable on $[a,b]$.

5. State and prove Newton's iterative method for finding roots. If

$$g(x) = x^3 - x^2 + x - 1, \quad x_1 = 1.01$$

and

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \quad n = 1, 2, \ldots ,$$

show that $x_n \to 1$ as $n \to \infty$.

6. Indicate how the approximation to $\int_a^b f(x)\,dx$, called Simpson's Rule, is obtained. If the fourth derivative $f^{(4)}(x)$ exists and is continuous on $[a,b]$, give an overestimate for the error incurred.

In estimating $\int_0^1 e^{-x^2}\,dx$, show that a subdivision of $[0,1]$ into 10 equal parts will be enough to produce an error of at most 10^{-4}.

CONTINUED

7. (i) Suppose F and G are differentiable functions on $[a, b]$ and $F' = f$, $G' = g$ where f and g are Riemann integrable on $[a, b]$. Show that

$$\int_a^b F(x)\, g(x)\, dx = F(b)\, G(b) - F(a)\, G(a) - \int_a^b f(x)\, G(x)\, dx.$$

Evaluate $\int_0^{\frac{\pi}{2}} x \cos^2 x dx$.

(ii) Suppose that $\int_a^b f(x)\, dx$ exists and $x = g(t)$ where

(a) $a = g(c)$, $b = g(d)$,

(b) $g'(t)$ exists and is continuous on $[c, d]$,

(c) $f(x)$ is continuous on $[a, b]$.

Show that

$$\int_a^b f(x)\, dx = \int_c^d f(g(t))\, g'(t)\, dt.$$

Evaluate $\int_2^e \frac{1}{x(\log x)^2} dx$; where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

END OF PAPER